II Local Theory

we will see that contact structures "locally" look the same. Most of these results will follow from

Thm 1:

let M be a closed oriented 3-manifold
$N \subset M$ a compact set
suppose ξ_0, ξ_1 are contact structures on M with

$$\xi_0|_N = \xi_1|_N$$

then there is a neighborhood U of N such that the identity map on M is isotopic, rel N, to a map that is a contactomorphism when restricted to U

we will prove this later but now we consider some of its consequenses

Thm 2 (Darboux):

let (M, ξ) be a contact manifold
Any point $p \in M$ has a neighborhood U that is contactomorphic to a neighborhood V of the origin in $(\mathbb{R}^3, \xi_{std})$ ker$(dz - y\,dx)$

Proof: we first find

1) neighborhood U' of $p \in M$,
2) " " V' of origin $\in \mathbb{R}^3$, and
3) diffeomorphism $\phi: U' \longrightarrow V'$

such that $d\phi_p(\xi) = \xi_{std}$

there are several ways to do this

1) take any diffeo ϕ' (say from a coordinate chart) and find a linear diffeo $A : \mathbb{R}^3 \to \mathbb{R}^3$ sending $d\phi'_p(\xi)$ to ξ_{std}, then $\phi = A \circ \phi'$

2) identify T_pM with $T_{(0,0,0)}\mathbb{R}^3$ by a linear map sending ξ to ξ_{std}

then use the exponential map (using some Riemannian metrics on M and $\mathbb{R}^3$)

$\xi' = d\phi(\xi)$ is a contact structure on V'

$\xi' = \xi_{std}$ at $(0,0,0)$

so Th$^{\underline{m}}$ 1 says there is an isotopy of $id : V' \to V'$ to $f : V' \to$ in V' such that

$f((0,0,0)) = (0,0,0)$ and

$df(\xi') = \xi_{std}$ on some nbhd V of $(0,0,0)$ in V'

$\therefore d(f \circ \phi)(\xi) = \xi_{std}$

Remark: So unlike Riemannian geometry, contact structures have no local invariants, making them "closer to" topology

a curve C in (M,ξ) is called <u>transverse</u> if

$$T_x C \pitchfork \xi_x \quad \text{for all } x \in C$$

if C is a closed curve it is called a <u>transverse knot</u>

Th$^{\underline{m}}$3:

$\boxed{\text{any two transverse knots have contactomorphic neighborhoods}}$

Proof: let $C_i \subset (M_i, \xi_i)$ be transverse knots for $i=0,1$

as in the proof of Th$^{\underline{m}}$2 we only need to find a diffeomorphism from a neighborhood of C_0 to a neighborhood of C_1 taking ξ_0 along C_0 to ξ_1 along C_1 and then apply Th$^{\underline{m}}$1

to this end, take any diffeomorphism

$$f: C_0 \to C_1$$

let

$$\begin{tikzcd} F: T_{C_0}M \arrow[r] \arrow[d] & T_{C_1}M \arrow[d] \\ C_0 \arrow[r, "f"'] & C_1 \end{tikzcd}$$

be a bundle map that covers f and sends ξ_0 to ξ_1

use the exponential map to extend f to a neighborhood of C_0

Recall an arc A is called <u>Legendrian</u> if

$$T_x A \subset \xi_x \quad \text{for all } x \in A$$

if A is closed it is called a <u>Legendrian knot</u>

Th$^{\underline{m}}$4:

$\boxed{\text{any two Legendrian knots have contactomorphic neighborhoods}}$

Proof: just like proof of Th$^{\underline{m}}$3 exercise

so we understand contact structures in neighborhoods of <u>points</u> and some <u>curves</u>

what about <u>surfaces</u>?

let Σ be an oriented surface in (M,ξ) (recall M is oriented)

let $\ell_x = \xi_x \cap T_x\Sigma$ for all $x \in \Sigma$

(oriented by orientation on M, Σ, and ξ) recall we assume all contact strs oriented

note: ℓ_x is generically a line in $T_x\Sigma$ (when $\xi_x \pitchfork T_x\Sigma$)

but some points are <u>singular</u> $\xi_x = T_x\Sigma$

choose a vector field v s.t.

v spans ℓ_x when ℓ_x a line

$v = 0$ when $T_x\Sigma = \xi_x$

exercise: show v exists

let Σ_ξ be the flow lines of v

this is a singular foliation

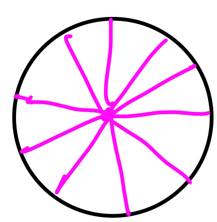
it is called the <u>characteristic foliation</u> of Σ

example: $\mathbb{R}^3$, $\xi = \ker(dz + r^2 d\theta) = \ker(dz + x\,dy - y\,dx)$

$f: D^2 \to \mathbb{R}^3 : (x,y) \mapsto (x,y,0)$

then $f^*\alpha = x\,dy - y\,dx = r^2 d\theta$

so $\ell_x = \ker f^*\alpha = \operatorname{span}\{r \frac{\partial}{\partial r}\}$



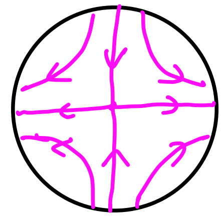
$g: D^2 \to \mathbb{R}^3 : (x,y) \mapsto (x,y,axy)$

then $g^*\alpha = x\,dy - y\,dx + a(x\,dy + y\,dx)$

$$= (a+1)x\,dy + (a-1)y\,dx$$

$v(x,y) = \begin{bmatrix}(1-a)y \\ (a+1)x\end{bmatrix}$ directs $\ker g^*\alpha$

so for $a > 1$ we have



Th$^{\underline{m}}$5:

$\Sigma^2 \subset M^3$

ξ_i a contact structure on M for $i = 0,1$

$\Sigma_{\xi_0} = \Sigma_{\xi_1}$

then there is a neighborhood U of Σ and an isotopy $\phi_t : M \to M$ such that

(1) $\phi_0 = id_M$

(2) ϕ_t is fixed on Σ

(3) $(\phi_1|_U)^*(\xi_0|_U) = \xi_1$

i.e. ϕ_1 a contactomorphism on U

Proof: this will follow from Th$^{\underline{m}}$1 if

$$\xi_0|_\Sigma = \xi_1|_\Sigma$$

but this is not necessarily true

need to isotop neighborhood of Σ in M

let $W = \Sigma \times (-\varepsilon, \varepsilon)$ be neighborhood of Σ in M

we construct an isotopy $\psi_t : W \to W$ such that

(1) ψ_t is fixed on Σ

(2) $\psi_0 = id$ on W

(3) $\psi_1^*(\xi_0(y,0)) = \xi_1(y,0)$

extend ψ_t to all of M (isotopy extention, how can we apply this?)

now can apply Th^m 1 to $(\psi_1)_* \xi_0$ and ξ_1 to get an isotopy

$\Phi_t : M \to M$ s.t. $(\Phi_1)_*((\psi_1)_* \xi_0) = \xi_1$ on nbhd of Σ

set $\phi_t = \Phi_t \circ \psi_t$ is desired isotopy.

we now construct ψ_t:

let $\xi_i = \ker \alpha_i$

$$\alpha_i|_\Sigma = \beta_i(y) + f_i(y)\, ds$$

for β_i a 1-form on Σ

f_i a function on Σ

$\Sigma_{\xi_i} = \ker \beta_i$

since $\Sigma_{\xi_0} = \Sigma_{\xi_1}$ we have a non-zero function $g : \Sigma \to \mathbb{R}$

such that $\beta_1 = g\beta_0$ (linear maps $\mathbb{R}^2 \to \mathbb{R}$ with same 1-dim kernel differ by a constant)

Claim: $g > 0$

to see this recall the contact condition for α_i on Σ

let $\alpha_i = \beta_i^s + f_i^s\, ds$ on W

$$d\alpha_i = d\beta_i^s + \frac{\partial \beta_i^s}{\partial s} \wedge ds + df_i^s \wedge ds$$

$$\alpha_i \wedge d\alpha_i = \underbrace{\cancel{\beta_i^s \wedge d\beta_i^s}}_{\text{3-form on } \Sigma} + \beta_i^s \wedge \frac{\partial \beta_i^s}{\partial s} \wedge ds + \beta_i^s \wedge df_i^s \wedge ds$$

$+ f_1^s \, ds \wedge d\beta_1^s$

$= \left(\beta_1^s \wedge \frac{\partial \beta_1^s}{\partial t} + \beta_1^s \wedge df_1^s + f_1 d\beta_1^s\right) \wedge ds$

so on Σ

$$\beta_1^0 \wedge \frac{\partial \beta_1^0}{\partial t} + \beta_1^0 \wedge df_1^0 + f_1 d\beta_1^0 > 0$$

at a singular point $f_1 \, d\beta_1^0 > 0$

and f_0, f_1 have same sign

since $\Sigma_{\xi_0} = \Sigma_{\xi_1}$

assume positive so $d\beta_1^0 > 0$

$$d\beta_1 = d(g\beta_0) = dg \wedge \beta_0 + g \, d\beta_0$$

at a singularity $d\beta_1 = g \, d\beta_0 \quad \therefore g > 0$

extend g to a positive function on all of M

replace α_0 by $g\alpha_0$

so $\alpha_0\big|_{T\Sigma} = \alpha_1\big|_{T\Sigma}$ i.e. $\beta_0 = \beta_1$ call it β

if $f_0 \neq 0$ then

$\psi : (y, s) = \left(y, \frac{f_1(y)}{f_0(y)} s\right)$ is well defined

$\psi : W \to W$

ψ fixes Σ

$\psi^* \alpha_0 = \beta + \frac{f_1(y)}{f_0(y)} f_0(y) \, ds = \alpha_1$ on Σ (so $s \, d\left(\frac{f_1}{f_0}\right)$ term is 0)

$\therefore \psi^* \xi_0 = \xi_1$ on Σ

(let ψ_t be obvious isotopy id to ψ)

in general f_0 will be 0 at some points

let $\Sigma_\delta = f_0^{-1}(|x| > \delta)$

note: if $(y,0) \in \Sigma$ is a singular point of Σ_ξ

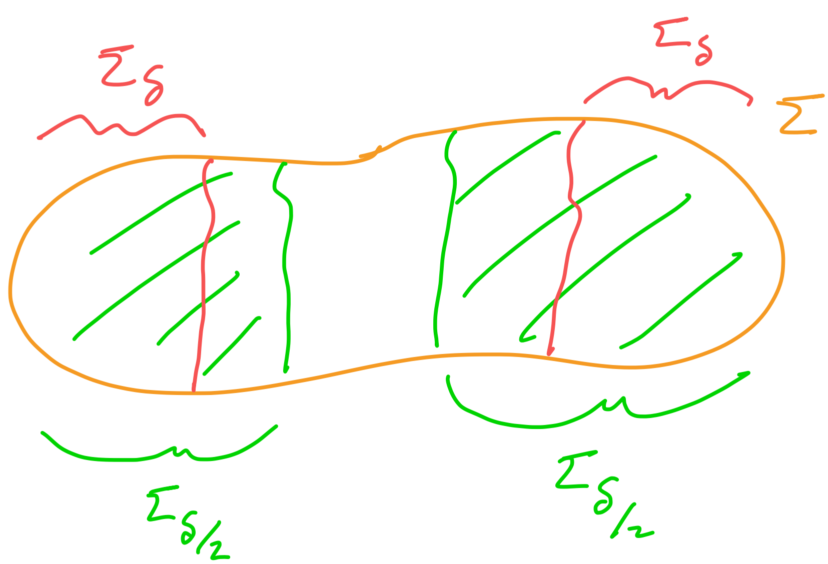
then $\beta_{(y,0)} = 0 \therefore f_1(y) \neq 0$

the set $f_1^{-1}(0)$ is compact so

$\delta_0 = \inf \{ f_0(x) \mid x \text{ singular} \}$

$\therefore$ for any $\delta < \delta_0$, all singularities are in Σ_δ

consider



define $\psi(y,t)$ as above on $\Sigma_{\delta/2}$

let $\bar{\Sigma} = \overline{\Sigma \setminus \Sigma_\delta}$ and $\tilde{\Sigma} = \bar{\Sigma} \cap \Sigma_{\delta/2}$

on $\bar{\Sigma}$ choose a vector field

$v = v_\Sigma + \frac{\partial}{\partial s}$ such that $v \in \xi_0$

$v_\Sigma \in T\Sigma$ (note $v \pitchfork \Sigma$)

note: v_Σ not in Σ_ξ

also choose w on $\bar{\Sigma}$ such that

(1) $d\psi(v) = w$ on $\tilde{\Sigma}$

(2) $w \pitchfork \Sigma$

(3) $w \in \xi_1$

we can write $w = w_\Sigma + h(y)\frac{\partial}{\partial s}$

where $w_\Sigma \in T\Sigma$, $h(y) > 0$

on $\tilde{\Sigma}$ we have

$$d\psi\left(v_\Sigma + \frac{\partial}{\partial s}\right) = w_\Sigma + h(y)\frac{\partial}{\partial s}$$

$$\parallel$$

$$v_\Sigma + \frac{f_1(y)}{f_0(y)}\frac{\partial}{\partial s}$$

since $\psi(y,s) = \left(y, \frac{f_1(y)}{f_0(y)} s\right)$ on $\tilde{\Sigma}$

$\therefore$ $v_\Sigma = w_\Sigma$ and $h(y) = \frac{f_1(y)}{f_0(y)}$ on $\tilde{\Sigma}$

(since $T_{(y,s)}M = T_y\Sigma \oplus \mathbb{R}_s$)

since v_Σ not in Σ, can assume

$$w_\Sigma = v_\Sigma$$

now set $\psi(y,s) = (y, h(y)s)$ on $\tilde{\Sigma}$

by construction $d\psi(\xi_0|_\Sigma) = \xi_1$ $\blacksquare$

in preparation for proving Th$^{\underline{m}}$ 1 we have

Th$^{\underline{m}}$ 6 (Gray's Th$^{\underline{m}}$):

let ξ_t, $t \in [0,1]$, be a family of contact structures on a manifold M that differ on a compact set $C \subset M$ (if M compact, $C = M$)

then there is an isotopy $\psi_t : M \to M$ such that

$$\psi_t^* \xi_0 = \xi_t \quad \text{and}$$

$$\psi_t = id \quad \text{off of } C$$

Remark: so any family of contact structures comes from an isotopy

i.e. an isotopy of contact structures ξ family in tangent bundle

can be "integrated" to

an isotopy of the manifold

this is not true for general plane fields

e.g. consider $\mathcal{F}_s$ a foliation of T^2 by lines of slope s

exercise: there is no isotopy of T^2 sending $\mathcal{F}_s$ to $\mathcal{F}_{s'}$ if $s \neq s'$

for 3D example consider $\mathcal{F} \times S^1 \subset T^2 \times S^1$

Proof: we will look for ψ_t as flow of a vector field X_t ← great way to find diffeos!

exercise: given ξ_t show there exists a family of 1-forms α_t such that

$$\xi_t = \ker \alpha_t$$

we want to find ψ_t that satisfy

$$\psi_t^* \alpha_t = \lambda_t \alpha_0 \quad \text{for } \lambda_t \neq 0 \text{ functions on } M$$

assuming ψ_t flow of X_t, let's compute

$$\frac{d}{dt}\left(\varphi_t^*\alpha_t\right) = \lim_{h\to 0} \frac{\varphi_{t+h}^*\alpha_{t+h} - \varphi_t^*\alpha_t}{h}$$

$$= \lim_{h\to 0} \frac{\varphi_{t+h}^*\alpha_{t+h} - \varphi_{t+h}^*\alpha_t + \varphi_{t+h}^*\alpha_t - \varphi_t^*\alpha_t}{h}$$

$$= \lim_{h\to 0} \varphi_{t+h}^*\left(\frac{\alpha_{t+h} - \alpha_t}{h}\right) + \lim_{h\to 0} \varphi_t^*\left(\frac{\varphi_h^*\alpha_t - \alpha_t}{h}\right)$$

$$= \varphi_t^* \frac{d\alpha_t}{dt} + \varphi_t^* \mathcal{L}_{X_t}\alpha_t$$

$$= \varphi_t^*\left(\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t}\alpha_t\right)$$

Lie derivative

we want

$$\varphi_t^*\left(\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t}\alpha_t\right) = \frac{d\lambda_t}{dt}\alpha_0 = \frac{d\lambda_t}{dt}\frac{1}{\lambda_t}\varphi_t^*\alpha_t$$

let $h_t = \frac{d}{dt}(\log\lambda_t)\circ\varphi_t^{-1}$ so we get

$$\varphi_t^*\left(\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t}\alpha_t\right) = \varphi_t^*(h_t\alpha_t)$$

if we can choose $X_t \in \xi_t$, then $\iota_{X_t}\alpha_t = 0$

recall Cartan magic formula

$$\mathcal{L}_{X_t}\alpha_t = d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t$$

so we get

$$\frac{d\alpha_t}{dt} + \iota_{X_t}d\alpha_t = h_t\alpha_t \quad (*)$$

we want to solve for X_t

α_t is given, but what about h_t?

is that another unknown?

recall the Reeb vector field v_{α_t} of α_t is the unique vector field that satisfies

$$\alpha_t(v_{\alpha_t}) = 1$$

$$\iota_{v_{\alpha_t}} d\alpha_t = 0$$

plug v_{α_t} into (*) to get

$$\frac{d\alpha_t}{dt}(v_{\alpha_t}) = h_t \quad (\#)$$

so h_t determined by α_t and (*) becomes

$$\iota_{X_t} d\alpha_t = h_t \alpha_t - \frac{d\alpha_t}{dt} \qquad (**)$$

to solve this equation we need a detour

let

$$(\Lambda^1_{\alpha_t})_x = \{ \beta \in T_x^* M \text{ s.t. } \beta(v_t) = 0 \}$$

so

$$\Lambda^1_{\alpha_t} = \coprod_{x \in M} (\Lambda^1_{\alpha_t})_x \subset T^* M$$

is a subbundle with 2D fiber

set $\Omega^1_{\alpha_t} = \Gamma(\Lambda^1_{\alpha_t}) =$ 1-forms vanishing on v_t

note:

$$\begin{aligned} (\xi_t)_x &\longrightarrow (\Lambda^1_{\alpha_t})_x \\ v &\longmapsto \iota_v d\alpha_t \end{aligned}$$

is an isomorphism since

- note $(\iota_v d\alpha_t)(v_{\alpha_t}) = d\alpha_t(v, v_{\alpha_t}) = 0$
  so $\iota_v d\alpha_t \in (\Lambda^1_{\alpha_t})_x$
- clearly linear
- if $\iota_w d\alpha_t = 0$, then $d\alpha_t(w, v) = 0$
  for all $v \in (\xi_t)_x$ $\therefore$ $w = 0$
  since $d\alpha_t$ non-degenerate on ξ_t

$\therefore$ map injective

- both 2D vector spaces

thus the map $\Gamma(\xi_t) \to \Omega^1_{\alpha_t}$
$$v \longmapsto \iota_v d\alpha_t$$
is an isomorphism

<u>note:</u> $(h_t \alpha_t - \frac{d\alpha_t}{dt})(v_{\alpha_t}) = h_t - \frac{d\alpha_t}{dt}(v_{\alpha_t}) = 0$ by $(\mp)$

so $\exists!$ vector field X_t s.t. $\iota_{X_t} d\alpha_t = h_t \alpha_t - \frac{d\alpha_t}{dt}$

by construction flow of X_t gives Ψ_t (check this if not clear)

lastly, where ξ_t agree we can choose α_t constant in t

$\therefore \frac{d\alpha_t}{dt} = 0$ and $h_t = 0$ so $X_t = 0$

and hence Ψ_t constant ▨

<u>Proof of Th^{m} 1:</u> recall we have ξ_0 and ξ_1 on M

and a compact set $N \subset M$ on which $\xi_0 = \xi_1$

let $\alpha_t = (1-t)\alpha_0 + t\alpha_1$

$\xi_t = \ker \alpha_t$ independent of t on N

$d\alpha_t = (1-t)\, d\alpha_0 + t\, d\alpha_1$

$d\alpha_0, d\alpha_1$ both area forms on $\xi_t|_N$ (can assume give same orientation)

$\therefore d\alpha_t$ gives an area form on $\xi_t|_N$

so there is a neighborhood U' of N s.t.

$d\alpha_t|_{\xi_t} \neq 0$ on U'

repeat proof of Gray's Th$^{\underline{m}}$ to get a vector field
X_t whose flow would give ψ_t

since $X_t = 0$ on N a compact set
there is a sufficiently small
neighborhood U of N such that
ψ_t exists for $t \in [0,1]$ and

$\psi_t(U) \subset U'$